THE p-PERIODICITY OF THE GROUPS $\mathrm{GL}(n, O_S(K))$ AND $\mathrm{SL}(n, O_S(K))$

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§1. *Introduction.* 1.1. In this paper we investigate the p-periodicity of the S-arithmetic groups $G = \mathrm{GL}(n, O_S(K))$ and $G_1 = \mathrm{SL}(n, O_S(K))$ where $O_S(K)$ is the ring of S-integers of a number field K (*cf.* **[12, 13]**; S is a finite set of places in K including the infinite places). These groups are known to be virtually of finite (cohomological) dimension, and thus the concept of p-periodicity is defined; it refers to a rational prime p and to the p-primary component $\hat{H}^i(G, A, p)$ of the Farrell–Tate cohomology $\hat{H}^i(G, A)$ with respect to an arbitrary G-module A. We recall that $\hat{H}^i$ coincides with the usual cohomology H^i for all i above the virtual dimension of G, and that in the case of a finite group (*i.e.*, a group of virtual dimension zero) the $\hat{H}^i$, $i \in \mathbb{Z}$, are the usual Tate cohomology groups. The group G is called p-periodic if $\hat{H}^i(G, A, p)$ is periodic in i, for all A, and the smallest corresponding period is then simply called the p-period of G. If G has no p-torsion, the p-primary component of all its $\hat{H}^i$ is 0, and thus G is trivially p-periodic.

We shall determine the rational primes p for which the above S-arithmetic groups are p-periodic, and compute the value of the p-period.

Partial results in that direction have been obtained earlier **[3]**. The present procedure is simpler and yields complete answers.

1.2. Our method is based on the following fact. Let G be any group of virtually finite dimension, and N a torsion-free normal subgroup of finite index in G. If G/N is p-periodic with p-period m_p, then G itself is p-periodic with p-period dividing m_p (see Section 5). In the case of the S-arithmetic groups G and G_1 above we take for N or N_1, respectively, the principal congruence subgroup of G or G_1, with respect to a certain prime ideal P of $O_S(K)$. This prime ideal can be chosen in such a way that N and N_1 are torsion-free and that the absolute norm $\mathfrak{N}(P) = |O_S(K)/P| = q$ is a rational prime suitable for our purpose. Then

$$G_1/N_1 \cong \mathrm{SL}(n, \mathbb{F}_q) \subset G/N \subset \mathrm{GL}(n, \mathbb{F}_q).$$

Thus the task is reduced essentially to investigating the p-periodicity of the finite groups $\mathrm{GL}(n, \mathbb{F}_q)$ and $\mathrm{SL}(n, \mathbb{F}_q)$. It turns out (Section 4) that both these groups are p-periodic if $\frac{1}{2}n < h_p(q) \leqslant n$, where $h_p(q)$ is the order of the residue class of q in $(\mathbb{Z}/p\mathbb{Z})^*$; and that then the p-period is $2h_p(q)$.

The "suitable choice" of P is such that, in addition to rendering N and N_1 torsion-free, its norm $\mathfrak{N}(P) = q$ fulfills $h_q(p) = \phi_K(p)$, the degree over K of the p-th cyclotomic extension $K(\zeta_p)$ of K. It then follows that G and G_1 are p-periodic for $\frac{1}{2}n < \phi_K(p) \leqslant n$ with p-period dividing $2\phi_K(p)$.

1.3. The existence of such a prime ideal is guaranteed by a *number-theoretic lemma* which we formulate and prove in Section 2, in a slightly more general version than actually needed (Lemma 2.2).

Let p be an odd rational prime, and r a positive integer. There exist infinitely many prime ideals P in $O_S(K)$ such that $\mathfrak{N}(P)$ is a rational prime q whose residue class has order $\phi_K(p^r)$ in $(\mathbb{Z}/p^r\mathbb{Z})^*$.

This lemma is useful also for other applications, in particular, in computations concerning the projective class group of certain arithmetic groups (see [**7**]), and in connection with topological problems as mentioned in [**4**].

1.4. In order to obtain, for the appropriate rational primes p, the precise value of the p-period of the groups G and G_1 we exhibit certain finite subgroups; they are obtained as semi-direct products of the group of p-th roots of unity with the Galois group of $K(\zeta_p)$ over K. Since quite generally any subgroup of a p-periodic group is also p-periodic, with p-period dividing that of the group, we thus get lower bounds for the p-periods of G and G_1. It turns out that they agree with the upper bounds $2\phi_K(p)$ except for the special case $\mathrm{SL}(\phi_K(p), O_S(K))$. The final results (Theorems 5.2 and 5.4 with Remarks) are as follows.

The groups $\mathrm{GL}(n, O_S(K))$, $n > 0$, and $\mathrm{SL}(n, O_S(K))$, $n > 2$, are p-periodic for all rational primes p with $\frac{1}{2}n < \phi_K(p) \leqslant n$; the p-period is $2\phi_K(p)$ except for $\mathrm{SL}(\phi_K(p), O_S(K))$ where it is either $\phi_K(p)$ or $2\phi_K(p)$ depending on the number field K. For $\phi_K(p) \leqslant \frac{1}{2}n$ they are not p-periodic, and for $\phi_K(p) > n$ they have no p-torsion. The group $\mathrm{SL}(2, O_S(K))$ is periodic (*i.e.*, p-periodic for all p) with period 2 or 4.

§2. *The number-theoretic lemma.* 2.1. We consider an algebraic number field K and its ring of integers $O(K)$. Let $\mathfrak{N}(I)$ denote the absolute norm $|O(K)/I|$ of the ideal I in $O(K)$.

LEMMA 2.1. *Let p be an odd prime number and r a positive integer. There exist infinitely many prime ideals P of $O(K)$ such that $\mathfrak{N}(P) = q$ is a prime number whose residue class has order $\phi_K(p^r)$ in $(\mathbb{Z}/p^r\mathbb{Z})^*$.*

Proof. The Galois group $\mathrm{Gal}(K(\zeta_{p^r})/K)$ is cyclic of order $\phi_K(p^r)$; let σ be a generator, *i.e.* $\sigma(\zeta_{p^r}) = \zeta_{p^r}^s$ where the order of the residue class of s in $(\mathbb{Z}/p^r\mathbb{Z})^*$ is $\phi_K(p^r)$.

We shall use results and notations of [**11**], Chapters IV and V. We consider the following "modulus" m. Let m_∞ be the product of all real places of K, and $m_0 = p^r O(K)$, and $m = m_0 m_\infty$. Let $K_{m,1}$ be defined by

$$K_{m,1} = \{x/y\,;\, x, y \in O(K) \text{ with } xO(K) \text{ and } yO(K) \text{ relatively prime to } m_0 \text{ and } x/y \equiv 1 \bmod m\};$$

and I_K^m the subgroup of the ideal group of K generated by all prime ideals not dividing m_0. The Artin map

$$\phi : I_K^m \to \mathrm{Gal}(K(\zeta_{p^r})/K)$$

is surjective, and its kernel contains the image $i(K_{m,1})$ of the embedding of $K_{m,1}$ in the ideal group by the reciprocity law for $(K(\zeta_{p^r}), K, m)$. Take $J \in I_K^m$ such that $\phi(J) = \sigma$. Then $\phi^{-1}(\sigma) = J \ker \phi \supset Ji(K_{m,1})$. By the generalized Dirichlet theorem there are in $\phi^{-1}(\sigma)$ infinitely many prime ideals, even if we require them to be of relative degree 1 (over $\mathbb{Z}$).

Let P be such a prime ideal of $O(K)$. The Frobenius automorphism

$$\left(\frac{K(\zeta_{p^r})/K}{P}\right)$$

is equal to $\sigma \in \mathrm{Gal}\big(K(\zeta_{p^r})/K\big)$. Since the relative degree of P is 1, we have $O(K)/P \cong \mathbb{Z}/q\mathbb{Z}$ where q is the rational prime over which P lies ($P \cap \mathbb{Z} = q\mathbb{Z}$). The Frobenius automorphism

$$\left(\frac{\mathbb{Q}(\zeta_{p^r})/\mathbb{Q}}{q}\right)$$

is the restriction of σ to $\mathbb{Q}(\zeta_{p^r})$; *i.e.*,

$$\zeta_{p^r}^q = \sigma(\zeta_{p^r}) = \zeta_{p^r}^s,$$

whence $q \equiv s \bmod p^r$. Thus q has order $\phi_K(p^r)$ in $(\mathbb{Z}/p^r\mathbb{Z})^*$.

2.2. We now consider the ring of S-integers $O_S(K)$ in K. Let Σ be the set of all places of K and S a subset of Σ containing Σ^∞, the set of infinite places. Then

$$O_S(K) = \bigcap_{Q \in \Sigma - S} O_Q$$

where O_Q is the valuation ring of Q. Hence $O_S(K)$ is a Dedekind ring with quotient field K.

If S above is a finite set then (*cf.* **[12]** or **[13]**) $\mathrm{GL}\big(n, O_S(K)\big)$ is virtually of finite dimension.

LEMMA 2.2. *Let S be a finite set of places including Σ^∞. Then the assertion of Lemma* 2.1 *also holds for $O_S(K)$.*

Indeed, all the prime ideals P occurring in Lemma 2.1, except for finitely many of them, generate prime ideals $P' = PO_S(K)$ of $O_S(K)$, and $\mathfrak{N}(P') = |O_S(K)/P'| = |O(K)/P| = \mathfrak{N}(P)$.

§3. *Finite subgroups.* 3.1. *Notation.* R is an integrally closed domain of characteristic zero, K its field of quotients, ζ_m a primitive m-th root of unity in an algebraic closure of K, $\phi_K(m) = [K(\zeta_m) : K]$, $Z_m = \langle \zeta_m \rangle$ the group of all m-th roots of unity, $C_k = \langle t \rangle$ any multiplicative cyclic group of order k with generator t (m, k are arbitrary natural numbers).

Let p be a rational prime, and let $C_{\phi_K(p)}$ operate on Z_p through the isomorphism $C_{\phi_K(p)} \cong \mathrm{Gal}\big(K(\zeta_p)/K\big)$ which maps t to a generator σ of the Galois group.

PROPOSITION 3.1. *The semi-direct product $Z_p \rtimes C_{\phi_K(p)}$ is p-periodic with p-period $2\phi_K(p)$.*

Proof. Obviously Z_p is a p-Sylow subgroup of $G = Z_p \rtimes C_{\phi_K(p)}$. Since it is cyclic, G is p-periodic (*cf.* [**8**, Chap. XII]). The p-period is given (*cf.* [**14**]) by $2|N_G(Z_p)/C_G(Z_p)|$ where N_G denotes the normalizer, C_G the centralizer in G. Now $N_G(Z_p) = G$ and $C_G(Z_p) = Z_p$, and hence the p-period is $2\phi_K(p)$.

3.2. The group in Proposition 3.1 can be embedded in $\mathrm{GL}(\phi_K(p), R)$, as follows. Since the irreducible polynomial in $K[x]$ of ζ_p is of degree $\phi_K(p)$ and has coefficients in R, the R-module $R[\zeta_p]$ is free with basis $1, \zeta_p, \ldots, \zeta_p^{\phi_K(p)-1}$. We can thus identify $\mathrm{GL}(\phi_K(p), R)$ with the group of R-module automorphisms $\mathrm{Aut}_R R[\zeta_p]$. Multiplication μ_{ζ_p} with ζ_p is an element of that group, and so is any element σ^s of $\mathrm{Gal}(K(\zeta_p)/K)$ if restricted to $R[\zeta_p]$.

We consider the subgroup $S = \{\mu_{\zeta_p}^r \sigma^s;\ 0 \leqslant r < p,\ 0 \leqslant s < \phi_K(p)\}$ of $\mathrm{Aut}_R R[\zeta_p]$. The map $Z_p \rtimes C_{\phi_K(p)} \to S$ given by $\zeta_p \mapsto \mu_{\zeta_p}$, $t \mapsto \sigma$ is easily seen to be an isomorphism. Thus $Z_p \rtimes C_{\phi_K(p)}$ is realized as a subgroup of $\mathrm{GL}(\phi_K(p), R)$, and therefore also of $\mathrm{GL}(n, R)$ for all $n \geqslant \phi_K(p)$.

THEOREM 3.2. *For a rational prime p with $\phi_K(p) \leqslant n$ the group $\mathrm{GL}(n, R)$ contains a finite subgroup which is p-periodic with p-period $2\phi_K(p)$.*

3.3. We now turn to the special linear groups over R. Since $\mathrm{SL}(n, R)$ contains $\mathrm{GL}(n-1, R)$ as a subgroup ($n > 1$) there is, for all p with $\phi_K(p) < n$, a finite subgroup in $\mathrm{SL}(n, R)$ which is p-periodic with p-period $2\phi_K(p)$. Some special arguments are needed in the case where $\phi_K(p) = n$ (> 1).

We can identify $\mathrm{SL}(\phi_K(p), R)$ with the subgroup $\mathrm{Aut}_R R[\zeta_p]_1$ of $\mathrm{Aut}_R R[\zeta_p]$ consisting of all automorphisms with determinant 1. The determinant of μ_{ζ_p} is a p-th root of 1 in K and hence $= 1$ since $\phi_K(p) > 1$. As for the generator σ of $\mathrm{Gal}(K(\zeta_p)/K)$, it has determinant $(-1)^{\phi_K(p)-1}$, indeed σ can be viewed as a cyclic permutation of a suitable basis of $K(\zeta_p)$ over K. Thus for odd $\phi_K(p) > 1$ the group S above actually lies in $\mathrm{Aut}_R R[\zeta_p]_1$. If $\phi_K(p)$ is even, $S_1 = S \cap \mathrm{Aut}_R R[\zeta_p]_1$ has index 2 in S; this group S_1 is p-periodic with p-period $\phi_K(p)$.

If $\phi_K(p)$ is even there are, however, also cases where one can have in $\mathrm{Aut}_R R[\zeta_p]_1$ a finite p-periodic subgroup S_2 with p-period $2\phi_K(p)$. This is so if there exists in $R[\zeta_p]$ a unit u with relative norm $\mathfrak{N}_{K(\zeta_p)/K}(u) = -1$. Indeed let again μ_u be multiplication in $R[\zeta_p]$ by u. This automorphism has determinant -1; thus $\mu_u\sigma$ has determinant 1 and generates in $\mathrm{Aut}_R R[\zeta_p]_1$ a cyclic subgroup of order $2\phi_K(p)$ (since $(\mu_u\sigma)^{\phi_K(p)} = -\text{identity}$). We put

$$S_2 = \{\mu_\zeta^r(\mu_u\sigma)^s,\quad 0 \leqslant r < p,\quad 0 \leqslant s < 2\phi_K(p)\}.$$

This subgroup of $\mathrm{Aut}_R R[\zeta_p]_1$ is isomorphic to $Z_p \rtimes C_{2\phi_K(p)}$ where the generator t of $C_{2\phi_K(p)}$ acts on Z_p through $t \mapsto \sigma$. The computation analogous to that in the proof of Proposition 3.1 shows that S_2 is p-periodic with p-period $2\phi_K(p)$.

In summary we have

THEOREM 3.3. (a) *For all p with $\phi_K(p) < n$, and for $\phi_K(p) = n$ if $\phi_K(p)$ is odd > 1, the group $\mathrm{SL}(n, R)$ contains a finite subgroup which is p-periodic with p-period $2\phi_K(p)$.*

(b) *If $\phi_K(p)$ is even, then $\mathrm{SL}(\phi_K(p), R)$ contains a finite subgroup which is p-periodic with p-period $\phi_K(p)$. If there is in $R[\zeta_p]$ a unit u with $\mathfrak{N}_{K(\zeta_p)/K}(u) = -1$, there exists even a finite subgroup with p-period $2\phi_K(p)$.*

§4. *The p-periodicity of* $\mathrm{GL}(n, \mathbb{F}_q)$ *and* $\mathrm{SL}(n, \mathbb{F}_q)$. 4.1. As usual $\mathbb{F}_{q^n}$ denotes the field of q^n elements; we recall that

$$|\mathrm{GL}(n, \mathbb{F}_q)| = q^{n(n-1)/2} \prod_{i=1}^{n} (q^i - 1) = (q-1)|\mathrm{SL}(n, \mathbb{F}_q)|.$$

Let p and q be different rational primes. We denote by $h_p(q)$ the order of the residue class of q in $(\mathbb{Z}/p\mathbb{Z})^*$. If $h = h_p(q)$ then p divides $q^h - 1$ but none of the other factors in $|\mathrm{GL}(h, \mathbb{F}_q)|$. Let p^a be the highest power of p dividing $q^h - 1$, *i.e.*, dividing $|\mathrm{GL}(h, \mathbb{F}_q)|$, and let S_p be a p-Sylow subgroup of $\mathrm{GL}(h, \mathbb{F}_q)$.

PROPOSITION 4.1. *The group S_p is cyclic; the centralizer of S_p in $\mathrm{GL}(h, \mathbb{F}_q)$ has index h in the normalizer.*

Proof. We write G for $\mathrm{GL}(h, \mathbb{F}_q)$ and identify G with the group of $\mathbb{F}_q$-vector space automorphisms of $\mathbb{F}_{q^h}$. For $x \in \mathbb{F}_{q^h}^*$ let μ_x be multiplication with x in $\mathbb{F}_{q^h}$; it is an element of $G = \mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_{q^h})$. Let g be a generator of the cyclic group $\mathbb{F}_{q^h}^*$ and $f = g^{(q^h-1)/p^a}$. Then $\mu_f \in G$ is of order p^a and generates a p-Sylow subgroup S_p of G.

To prove the second part we show that $N_G(S_p)/C_G(S_p)$ is isomorphic to $\mathrm{Gal}(\mathbb{F}_{q^h}/\mathbb{F}_q)$ and hence of order h. Indeed $\mathrm{Gal}(\mathbb{F}_{q^h}/\mathbb{F}_q)$ is contained in G and one easily checks (*cf.* [**6**], Lemma 3.2 or [**10**], Chap. II, §7) that

$$N_G(S_p) = \{\mu_x \gamma;\quad x \in \mathbb{F}_{q^h}^*,\quad \gamma \in \mathrm{Gal}(\mathbb{F}_{q^h}/\mathbb{F}_q)\},$$

and

$$C_G(S_p) = \{\mu_x;\quad x \in \mathbb{F}_{q^h}^*\}.$$

Thus $C_G(S_p)$ is the kernel of the obvious map $N_G(S_p) \to \mathrm{Gal}(\mathbb{F}_{q^h}/\mathbb{F}_q)$ and the assertion follows.

4.2. From Proposition 4.1 it follows that $\mathrm{GL}(h, \mathbb{F}_q)$, $h = h_p(q)$, is p-periodic with p-period $2h$. We shall show that the same holds for $\mathrm{GL}(n, \mathbb{F}_q)$ if $\frac{1}{2}n < h \leqslant n$.

Let $B \in \mathrm{GL}(h, \mathbb{F}_q)$ be a matrix of order p^a, generating S_p. Then

$$B' = \begin{pmatrix} B & 0 \\ 0 & E \end{pmatrix},$$

where E is the $(n-h) \times (n-h)$ unit matrix, has order p^a in $\mathrm{GL}(n, \mathbb{F}_q)$. The assumption $n < 2h$ guarantees that p^a is the highest power of p dividing $|\mathrm{GL}(n, \mathbb{F}_q)|$. Thus B' generates a cyclic p-Sylow subgroup S'_p of $\mathrm{GL}(n, \mathbb{F}_q)$. The normalizer of S'_p is given by the matrices

$$\left\{\begin{pmatrix} N & 0 \\ 0 & D \end{pmatrix};\quad N \in N_{\mathrm{GL}(h, \mathbb{F}_q)}(S_p),\quad D \in \mathrm{GL}(n-h, \mathbb{F}_q)\right\},$$

and similarly for the centralizer of S'_p. It immediately follows that the index of the centralizer of S'_p in the normalizer is again h; thus the p-period of $\mathrm{GL}(n, \mathbb{F}_q)$ is $2h$.

4.3. The remaining cases $n < h$ and $n \geqslant 2h$ are easy.

If $n < h = h_p(q)$ then p does not divide $|\mathrm{GL}(n, \mathbb{F}_q)|$; *i.e.*, $\mathrm{GL}(n, \mathbb{F}_q)$ has no p-torsion.

If $n \geqslant 2h$ we take an embedding

$$\mathrm{GL}(h, \mathbb{F}_q) \times \mathrm{GL}(h, \mathbb{F}_q) \subset \mathrm{GL}(2h, \mathbb{F}_q) \subset \mathrm{GL}(n, \mathbb{F}_q).$$

Since p divides $|\mathrm{GL}(h, \mathbb{F}_q)|$ there is a cyclic subgroup C_p in $\mathrm{GL}(h, \mathbb{F}_q)$. Thus $\mathrm{GL}(n, \mathbb{F}_q)$ contains a subgroup $C_p \times C_p$ and can therefore not be p-periodic.

4.4. We now turn to the group $\mathrm{SL}(n, \mathbb{F}_q)$, first for $n \geqslant 3$, and show that all the p-periodicity statements for $\mathrm{GL}(n, \mathbb{F}_q)$ above also hold for $\mathrm{SL}(n, \mathbb{F}_q)$, $n \geqslant 3$.

We may, of course, assume q odd. So $\mathrm{SL}(n, \mathbb{F}_q)$, being a subgroup of $G = \mathrm{GL}(n, \mathbb{F}_q)$, is p-periodic for $\frac{1}{2}n < h \leqslant n$, $h = h_p(q)$, with p-period dividing $2h$. The crucial case is again $\mathrm{SL}(h, \mathbb{F}_q)$; by assumption $h > \frac{1}{2}n > 1$.

We write G_1 for $\mathrm{SL}(h, \mathbb{F}_q)$ and identify G_1 with $\mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_{q^h})_1$ where the index 1 refers to determinant 1. With notations as in 4.1 the automorphism μ_f has determinant 1 since p does not divide $q-1 = |\mathbb{F}_q^*|$. Thus the cyclic group S_p generated by μ_f lies in G_1. Its normalizer is $N_G(S_p) \cap G_1$ and its centralizer is $C_G(S_p) \cap G_1$.

For the generator g of $\mathbb{F}_{q^h}^*$ the determinant $\det \mu_g$ is $g^{(q^h-1)/(q-1)} \in \mathbb{F}_q^*$; and for the generator $\sigma \in \mathrm{Gal}(\mathbb{F}_{q^h}/\mathbb{F}_q)$, $\det \sigma = (-1)^{h-1} \in \mathbb{F}_q^*$ since σ may be viewed as a cyclic permutation of order h. Thus the elements $\mu_x \gamma$, $x \in \mathbb{F}_{q^h}^*$, $\gamma \in \mathrm{Gal}(\mathbb{F}_{q^h}/\mathbb{F}_q)$, of $N_G(S_p)$ have determinant 1 in the following cases.

If h is odd: $x = g^{r(q-1)}$, $\quad 0 \leqslant r < (q^h-1)/(q-1)$; $\quad \gamma = \sigma^s$, $\quad 0 \leqslant s < h$.

If h is even: $x = g^{r(q-1)}$, $\quad 0 \leqslant r < (q^h-1)/(q-1)$; $\quad \gamma = \sigma^{2s}$, $\quad 0 \leqslant s < \frac{1}{2}h$,

and $\quad x = g^{r(q-1)+\frac{1}{2}(q-1)}$, $\quad 0 \leqslant r < (q^h-1)/(q-1)$; $\quad \gamma = \sigma^{2s+1}$, $\quad 0 \leqslant s < \frac{1}{2}h$.

The elements $\mu_x, x \in \mathbb{F}_{q^h}^*$, of $C_G(S_p)$ have determinant 1, if, and only if, $x = g^{r(q-1)}, 0 \leqslant r < (q^h-1)/(q-1)$. A simple count shows that the index of the centralizer in the normalizer is h; hence the p-period of $\mathrm{SL}(n, \mathbb{F}_q)$, $n \geqslant 3$, is $2h$.

4.5. We summarize as follows.

THEOREM 4.2. *Let p and q be different prime numbers, and $h = h_p(q)$ the order of q in $(\mathbb{Z}/p\mathbb{Z})^*$. If $\frac{1}{2}n < h \leqslant n$, then the groups $\mathrm{GL}(n, \mathbb{F}_q)$, $n \geqslant 1$, and $\mathrm{SL}(n, \mathbb{F}_q)$, $n \geqslant 3$, are p-periodic with p-period $2h$.*

Remark 4.3. (a) For $\frac{1}{2}n \geqslant h = h_p(q)$ the groups in Theorem 4.2 are not p-periodic.

(b) For $n < h$ they have no p-torsion.

Indeed, (a) is proved in 4.3 for $\mathrm{GL}(n, \mathbb{F}_q)$. If $h \geqslant 2$ $(n \geqslant 4)$, then p does not divide $q-1 = |\mathbb{F}_q^*|$, and the subgroup $C_p \times C_p$ mentioned in 4.3 actually lies in $\mathrm{SL}(n, \mathbb{F}_q)$. If $h = 1$ a special argument is needed for $\mathrm{SL}(n, \mathbb{F}_q)$, $n \geqslant 3$. In that case p divides $q-1$; let $x \in \mathbb{F}_{q-1}^*$ be of order p. The matrices

$$\begin{pmatrix} x^r & 0 & 0 \\ 0 & x^s & 0 \\ 0 & 0 & x^{-r-s} \end{pmatrix}$$

with $0 \leqslant r, s < p$ constitute a subgroup of $\mathrm{SL}(3, \mathbb{F}_q)$ isomorphic to $C_p \times C_p$. Thus $\mathrm{SL}(n, \mathbb{F}_q)$, $n \geqslant 3$, is not p-periodic in that case. The result (b) is proved in §4.3.

Remark 4.4. $\mathrm{SL}(2, \mathbb{F}_q)$ is well known to be p-periodic for all p. The q-period is $q-1$ for odd q, and 2 for $q = 2$. For p dividing q^2-1 the p-period is 4.

§5. *Finite quotients. Main results.* 5.1. We now turn to the groups $G = \mathrm{GL}(n, O_S(K))$ and $G_1 = \mathrm{SL}(n, O_S(K))$ described in Section 1. K is a number field, S a finite set of places including the infinite places, $O_S(K)$ the ring of S-integers of K.

We choose, by virtue of Lemma 2.2, a prime ideal P of $O_S(K)$ such that $\mathfrak{N}(P)$ is a prime number $q > 2^{[K:\mathbb{Q}]}$, and that $h_p(q) = \phi_K(p)$; p is a given prime number and $h_p(q)$ is the order of q in $(\mathbb{Z}/p\mathbb{Z})^*$. Then $O_S(K)/P \cong \mathbb{F}_q$, and reducing all matrix entries modulo P yields canonical maps $\psi: G \to \mathrm{GL}(n, \mathbb{F}_q)$ and $\psi_1: G_1 \to \mathrm{SL}(n, \mathbb{F}_q)$. Their kernels are the respective congruence subgroups modulo P, $N \subset G$ and $N_1 \subset G_1$. Due to the choice of P they are torsion-free (*cf.* **[2]**, for example). The map ψ_1 is known to be surjective ([**1**], p. 267), *i.e.*, we have

$$G_1/N \cong \mathrm{SL}(n, \mathbb{F}_q) \subset \mathrm{Im}\,\psi \subset \mathrm{GL}(n, \mathbb{F}_q).$$

As shown in Section 4 both $\mathrm{SL}(n, \mathbb{F}_q)$ and $\mathrm{GL}(n, \mathbb{F}_q)$ are p-periodic with p-period $2h_p(q) = 2\phi_K(p)$ for all prime numbers p with $\frac{1}{2}n < \phi_K(p) \leqslant n$; thus the same holds for G/N and G_1/N_1.

Proposition 5.1. *There exists a prime ideal P in $O_S(K)$ such that the congruence subgroups modulo P, $N \subset G$ and $N_1 \subset G_1$, are torsion-free and such that the finite quotients G/N and G_1/N_1 are p-periodic with p-period $2\phi_K(p)$ for all p with $\frac{1}{2}n < \phi_K(p) \leqslant n$.*

5.2. We now invoke a general result concerning the Farrell–Tate cohomology of a group G of virtually finite dimension. Let N be a torsion-free normal subgroup of finite index in G such that G/N is p-periodic with p-period m_p; then G itself is p-periodic with p-period dividing m_p. In the case, where G admits a projective resolution which is finitely generated in all dimensions, this result is proved in **[2]** using the construction of a complete resolution for G from a complete resolution for G/N, *cf.* [**2**] or [**9**]. Actually the result holds without any finiteness condition (see [**5**]); in the present context this generality is not needed since the above finiteness condition holds for $\mathrm{GL}(n, O_S(K))$ and $\mathrm{SL}(n, O_S(K))$ according to Borel–Serre (see [**13**], *e.g.*).

It thus follows that our groups G and G_1 are p-periodic for the appropriate prime numbers p, and that the p-period divides $2\phi_K(p)$.

5.3. To obtain the precise value of the p-period we use the finite subgroups constructed in Section 3. By Theorems 3.2 and 3.3 the groups $G = \mathrm{GL}(n, O_S(K))$, $n \geqslant \phi_K(p)$, and $G_1 = \mathrm{SL}(n, O_S(K))$, $n > \phi_K(p)$ contain a finite subgroup which has p-period $2\phi_K(p)$. Thus, for $\frac{1}{2}n < \phi_K(p) \leqslant n$ (or $< n$ respectively) the p-period of $\mathrm{GL}(n, O_S(K))$ and $\mathrm{SL}(n, O_S(K))$ respectively is equal to $2\phi_K(p)$. The case $\mathrm{SL}(\phi_K(p), O_S(K))$ is discussed in 5.4 below.

THEOREM 5.2. *The groups* $\mathrm{GL}(n, O_S(K))$, $\frac{1}{2}n < \phi_K(p) \leqslant n$, *and* $\mathrm{SL}(n, O_S(K))$, $\frac{1}{2}n < \phi_K(p) < n$, *are p-periodic with p-period* $2\phi_K(p)$.

Remark 5.3. The groups $\mathrm{GL}(n, O_S(K))$ and $\mathrm{SL}(n, O_S(K))$ have p-torsion, if, and only if, $\phi_K(p) \leqslant n$, see [**3**]. Using this fact one can, if $n \geqslant 2\phi_K(p)$, easily find a subgroup of these groups (for $\mathrm{SL}(n, O_S(K))$ assuming $n \geqslant 3$) isomorphic to $C_p \times C_p$. Therefore they are not p-periodic if $\frac{1}{2}n \geqslant \phi_K(p)$.

5.4. In the special case $\mathrm{SL}(\phi_K(p), O_S(K))$ all the above arguments remain valid except that Theorem 3.3 yields, in general, the two possibilities $\phi_K(p)$ or $2\phi_K(p)$ for the p-period. If $\phi_K(p)$ is odd and greater than one, the p-period is $2\phi_K(p)$, by Theorem 3.3(a). If $\phi_K(p)$ is even, the precise value depends on the norm map $\mathfrak{N}_{K(\zeta_p)/K}$. By Theorem 3.3(b) the period is again $2\phi_K(p)$, if there exists in $O_S(K)[\zeta_p]$ a unit u with $\mathfrak{N}_{K(\zeta_p)/K}(u) = -1$.

THEOREM 5.4. *The group* $\mathrm{SL}(\phi_K(p), O_S(K))$, $\phi_K(p) > 1$, *is p-periodic with p-period* $\phi_K(p)$ *or* $2\phi_K(p)$. *If* $\phi_K(p)$ *is odd or, more generally, if there is in* $O_S(K)[\zeta_p]$ *a unit with norm* -1 *over* K, *then the p-period is* $2\phi_K(p)$.

Remark 5.5. If there is no element in $K(\zeta_p)$ with norm -1 over K, then the p-period of $\mathrm{SL}(\phi_K(p), O_S(K))$ is $\phi_K(p)$. This follows from the computations in [**6**], Section 8. The condition is fulfilled, in particular, if K has an embedding in $\mathbb{R}$. Thus $\mathrm{SL}(p-1, \mathbb{Z})$, for example, is p-periodic with p-period $p-1$ (this case appears in [**3**] and is obtained by an entirely different method).

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